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Collocation-Continuation Technique for Solving Nonlinear Ordinary Boundary Value Problems

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Abstract—A new numerical technique is presented for solving the nonlinear ordinary boundary value problems. Theoretical and numerical results are presented. Comparison with Syam's results [1] will be given. These results indicate that our technique works more nicely and efficiently than his technique.

Keywords—Legendre polynomials, Nonlinear ordinary boundary value problems, Collocation and continuation methods.

1. INTRODUCTION

Syam [1] gave a numerical technique for solving nonlinear ordinary boundary value problems of the form

$$V''(x) + p(x)V(x)V'(x) + q(x)V(x) = f(x), x \in]-1,1[, \tag{1}$$

$$V(-1) = g_-, \qquad V(1) = g_+, \tag{2}$$

where $p(x)$, $q(x)$, and $f(x)$ are continuous functions on $[-1,1]$, and g_- and g_+ are constants. The basic idea of his technique involve the Chebyshev-Tau method to discretize problem (1),(2). The result was a nonlinear system of equations that is solved by an iterative method. However, this iterative method is depending on the initial guess which causes some problems. In this paper, we want to present a numerical technique for solving the same problem. Our idea is to apply the Legendre collocation method to discretize problem (1),(2) to get a nonlinear system. Then, we will use the continuation method to solve it. This method will not need any initial guess. So, we expect better results than the classical nonlinear solvers such as Newton or secant methods.

In Section 2, we present some definitions and fact that we use hereafter. In Section 3, we present our approach for solving problem (1),(2). Further, theoretical results are presented in

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the same section. Finally, in Section 4, we give some of our numerical experiments. In addition, analysis and conclusions will be presented.

2. PRELIMINARIES

The basic concept of this paper is the Legendre polynomials. For this reason, we study some of their properties.

DEFINITION 1. *The Legendre polynomials* $\{L_k(x) : k = 0, 1, \dots\}$ *are the eigenfunctions of the singular Sturm-Liouville problem*

$$\left(\left(1 - x^2\right) L_k'(x)\right)' + k(k+1)L_k(x) = 0, \qquad x \in [-1, 1].$$

Among the properties of the Legendre polynomials we list the following three properties:

$$\int_{-1}^{1} L_i(x)L_j(x)\,dx = \begin{cases} (i+0.5)^{-1}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases} \tag{3}$$

$$L_{i+1}(x) = \frac{2i+1}{i+1} x L_i(x) - \frac{i}{i+1} L_{i-1}(x), \tag{4}$$

for $i \geq 1$ and the endpoint relation

$$L_i(\pm 1) = (\pm 1)^i. \tag{5}$$

Suppose that $v(x) \in C^2[-1, 1]$ and $v^{(3)}(x)$ is a piecewise continuous function on the interval $[-1, 1]$. So, for the function $v(x)$, we can form the infinite Legendre expansion $v(x) = \sum_{i=0}^{\infty} v_i L_i(x)$. Then for

$$Qv(x) = \frac{d^s}{dx^s} v(x),$$

we have $\sum_{i=0}^{\infty} v_i^{(s)} L_i(x)$ converges uniformly on $[-1, 1]$ to $Qv(x)$, where $s = 1, 2$. The coefficients also satisfy the relations

$$v_i^{(1)} = (2i+1) \sum_{\substack{p=i+1 \\ p+i \text{ odd}}} v_p \tag{6}$$

and

$$v_i^{(2)} = (i+0.5) \sum_{\substack{p=i+2 \\ p+i \text{ even}}} [p(p+1) - i(i+1)] v_p. \tag{7}$$

For more details, see [2].

DEFINITION 2. *If* $\mathbf{x} = [x_1 x_2 \dots x_n]^\top$ *and* $\mathbf{y} = [y_1 y_2 \dots y_n]^\top$ *are* $n \times 1$ *real matrices, then the pointwise product* $\mathbf{x} \odot \mathbf{y}$ *of* $\mathbf{x}$ *and* $\mathbf{y}$ *is defined by*

$$\mathbf{x} \odot \mathbf{y} = [x_1 y_1 x_2 y_2 \dots x_n y_n]^\top,$$

where $\top$ means the transpose of the matrix.

It is easy to see that $\mathbf{x} \odot \mathbf{y} = D\mathbf{y}$, where $D = \mathrm{diag}(x_1, x_2, \dots, x_n)$ is an $n \times n$ diagonal matrix. Let $A = (a_{i,j})$ and $B = (b_{i,j})$ be two $n \times n$ matrices and let $H : \Re^n \to \Re^n$ be a function that is defined by $H(\mathbf{x}) = A\mathbf{x} \odot B\mathbf{x}$. Let $r_1, r_2, \dots, r_n$ be the rows of A and let $c_1, c_2, \dots, c_n$ be the columns of B. Thus, we can write $H(\mathbf{x})$ as

$$H(\mathbf{x}) = A\mathbf{x} \odot B\mathbf{x} = \Psi(\mathbf{x})B\mathbf{x},$$

where $\Psi(\mathbf{x}) = \mathrm{diag}(r_1\mathbf{x}, r_2\mathbf{x}, \dots, r_n\mathbf{x})$. Hence, $H'(\mathbf{x}) = \Psi(\mathbf{x})B + \Xi(\mathbf{x})$ where

$$\Xi(\mathbf{x}) = [\Lambda_1 B\mathbf{x}, \Lambda_2 B\mathbf{x}, \dots, \Lambda_n B\mathbf{x}],$$

and Λ_i is an $n \times n$ matrix such that

$$(\Lambda_i)_{jk} = \begin{cases} a_{ik}, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for all $i = 1 : n$. Thus, the closed form of $H'(\mathbf{x})$ is given by

$$H'(\mathbf{x}) = [(r_1\mathbf{x})c_1 + \Lambda_1 B\mathbf{x}, (r_2\mathbf{x})c_2 + \Lambda_2 B\mathbf{x}, \dots, (r_n\mathbf{x})c_n + \Lambda_n B\mathbf{x}]. \tag{8}$$

Finally, we want to state the following definition.

DEFINITION 3. *Let A be an $n \times (n+1)$ matrix with maximal rank. Then the Moore-Penrose inverse of A is defined by $A^+ = A^\top (AA^\top)^{-1}$.*

3. COLLOCATION-CONTINUATION TECHNIQUE

In this section, we want to present a numerical technique for solving problem (1),(2). We use the Legendre collocation method to discretize problem (1),(2). The result is a nonlinear system. The standard methods for solving the nonlinear systems, such as the Newton and the secant methods, need a good initial guess. Unfortunately, this is not possible in all of the cases. Hence, we look for a method for solving a nonlinear system without looking for a good initial guess. Thus, we can use the continuation method for this purpose. Approximate the solution $V(x)$ in terms of the Legendre polynomials as follows:

$$V_N(x) = \sum_{k=0}^{N+2} u_k L_k(x). \tag{9}$$

Thus, the first and the second derivatives of $V(x)$ can be approximated by

$$V_N'(x) = \sum_{k=0}^{N+1} u_k^{(1)} L_k(x) \tag{10}$$

and

$$V_N''(x) = \sum_{k=0}^{N} u_k^{(2)} L_k(x), \tag{11}$$

where $u_k^{(1)}$ and $u_k^{(2)}$ can be computed from equations (6) and (7), respectively. Therefore, for V_N, the residual is given by

$$R(V_N) = V_N''(x) + p(x)V_N(x)V_N'(x) + q(x)V_N(x) - f(x). \tag{12}$$

Orthogonalize the residual with respect to the Dirac delta functions as follows:

$$\langle R(V_N)\delta(x - x_j)\rangle = \int_{-1}^{1} R(V_N(x))\delta(x - x_j)\, dx = 0, \qquad \text{for } j = 0 : N, \tag{13}$$

where x_j are the collocation points. We choose the collocation points to be the roots of $L_{N+2}'(x)$. Therefore, equation (13) leads to the elementwise equation

$$V_N''(x_j) + p(x_j)V_N(x_j)V_N'(x_j) + q(x_j)V_N(x_j) = f(x_j), \qquad \text{for } j = 0 : N. \tag{14}$$

Using equations (9)–(11), we can rewrite equation (14) as

$$\begin{gathered}\sum_{k=0}^{N} u_k^{(2)} L_k(x_j) + p(x_j) \left(\sum_{k=0}^{N+2} u_k L_k(x_j)\right) \left(\sum_{k=0}^{N+1} u_k^{(1)} L_k(x_j)\right), \\ + q(x_j) \left(\sum_{k=0}^{N+2} u_k L_k(x_j)\right) = f(x_j),\end{gathered} \tag{15}$$

for $j = 0 : N$. Let $\mathbf{U} = [u_0 u_1 \dots u_{N+2}]^\top, \mathbf{U}^{(1)} = [u_0^{(1)} u_1^{(1)} \dots u_{N+1}^{(1)}]^\top, \mathbf{U}^{(2)} = [u_0^{(2)} u_1^{(2)} \dots u_N^{(2)}]^\top$, and $\mathbf{F} = [f(x_0) f(x_1) \dots f(x_N)]^\top$. Hence, we can rewrite equation (15) in the matrix form as

$$B_1 \mathbf{U}^{(2)} + B_2 \mathbf{U} \odot B_3 \mathbf{U}^{(1)} + B_4 \mathbf{U} = \mathbf{F}, \tag{16}$$

where B_1, B_2, B_3, and B_4 are $(N+1) \times (N+1), (N+1) \times (N+3), (N+1) \times (N+2)$ and $(N+1) \times (N+3)$ matrices, respectively, such that

$$B_{1_{i,j}} = L_{j-1}(x_{i-1}), \qquad \text{for } i = 1 : N+1, \quad j = 1 : N+1, \tag{17}$$

$$B_{2_{i,j}} = p(x_{i-1}) L_{j-1}(x_{i-1}), \qquad \text{for } i = 1 : N+1, \quad j = 1 : N+3, \tag{18}$$

$$B_{3_{i,j}} = L_{j-1}(x_{i-1}), \qquad \text{for } i = 1 : N+1, \quad j = 1 : N+2, \tag{19}$$

and

$$B_{4_{i,j}} = q(x_{i-1}) L_{j-1}(x_{i-1}), \qquad \text{for } i = 1 : N+1, \quad j = 1 : N+3. \tag{20}$$

From equations (6) and (7), there exist $(N+2) \times (N+3)$ and $(N+2) \times (N+3)$ matrices A_1 and A_2 such that

$$\mathbf{U}^{(1)} = A_1 \mathbf{U} \quad \text{and} \quad \mathbf{U}^{(2)} = A_2 \mathbf{U}. \tag{21}$$

Therefore, system (16) becomes

$$B_1 A_2 \mathbf{U} + B_2 \mathbf{U} \odot B_3 A_1 \mathbf{U} + B_4 \mathbf{U} = \mathbf{F}, \tag{22}$$

or

$$Q_1 \mathbf{U} + Q_2 \mathbf{U} \odot Q_3 \mathbf{U} = \mathbf{F}, \tag{23}$$

where $Q_1 = B_1 A_2 + B_4$, $Q_2 = B_2$, and $Q_3 = B_3 A_1$.

Now, we want to study the boundary conditions. From equation (5), one can see that

$$V_N(1) = \sum_{k=0}^{N+2} u_k L_k(1) = \sum_{k=0}^{N+2} u_k \tag{24}$$

and

$$V_N(-1) = \sum_{k=0}^{N+2} u_k L_k(-1) = \sum_{k=0}^{N+2} (-1)^k u_k. \tag{25}$$

From equations (2), (24), and (25), we get

$$g_+ = \sum_{k=0}^{N+2} u_k \quad \text{and} \quad g_- = \sum_{k=0}^{N+2} (-1)^k u_k. \tag{26}$$

Let $\mathbf{E}_1 = (1,\ 1,\ \dots, 1)$ and $\mathbf{E}_2 = (1,\ -1,\ \dots,\ (-1)^{N+2})$ be $1 \times (N+3)$ matrices. Then,

$$g_+ = \mathbf{E}_1 \mathbf{U} \quad \text{and} \quad g_- = \mathbf{E}_2 \mathbf{U}. \tag{27}$$

From systems (23) and (26), we obtain the following nonlinear system:

$$\begin{pmatrix} Q_1 \\ \mathbf{E}_1 \\ \mathbf{E}_2 \end{pmatrix} \mathbf{U} + \begin{pmatrix} Q_2 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{U} \odot \begin{pmatrix} Q_3 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix} \mathbf{U} = \begin{pmatrix} \mathbf{F} \\ g_+ \\ g_- \end{pmatrix}, \tag{28}$$

or

$$M\mathbf{U} + A\mathbf{U} \odot B\mathbf{U} = \mathbf{R}, \tag{29}$$

where

$$M = \begin{pmatrix} Q_1 \\ \mathbf{E}_1 \\ \mathbf{E}_2 \end{pmatrix}, \quad A = \begin{pmatrix} Q_2 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad B = \begin{pmatrix} Q_3 \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} \mathbf{F} \\ g_+ \\ g_- \end{pmatrix}.$$

The standard methods for solving the nonlinear system (29), such as the Newton and the secant methods, need a good initial guess which is not available. To avoid this difficulty, we look for another technique that does not depend on the initial guess. The promise technique is called the continuation method which is described as follows..

Let $W : \Re^{N+3} \to \Re^{N+3}$ be a function that is given by $W(\mathbf{U}) = M\mathbf{U} + A\mathbf{U} \odot B\mathbf{U} - \mathbf{R}$. We are looking for the zero of the function W. Define the function $G : \Re^{N+3} \to \Re^{N+3}$ by $G(\mathbf{U}) = M\mathbf{U}$ and define $H : \Re^{N+3} \times [0,1] \to \Re^{N+3}$ by

$$H(\mathbf{U}, \lambda) = \lambda W(\mathbf{U}) + (1-\lambda)G(\mathbf{U}) = M\mathbf{U} + \lambda(A\mathbf{U} \odot B\mathbf{U} - \mathbf{R}). \tag{30}$$

We should note that $H \in C^{\infty}(\text{domain}(H))$, $H(\mathbf{0},0) = \mathbf{0}$, and $\frac{\partial H}{\partial \mathbf{U}}(\mathbf{0},0) = M$. Since $M\mathbf{U} = R$ is the linear system that is produced when we apply the Legendre collocation method on the following problem:

$$v''(x) + q(x)v(x) = f(x), \quad x \in]-1,1[v(-1) = g_-, \quad \text{and} \quad v(1) = g_+ \tag{31}$$

and problem (31) has a unique solution, so the corresponding matrix M is a nonsingular matrix [2]. Hence, $\frac{\partial H}{\partial \mathbf{U}}(\mathbf{0},0)$ is a nonsingular matrix. Thus, it follows from the Implicit Function Theorem that there exists a smooth curve $C : J \to \Re^{N+3}$ for some open interval J containing zero such that $C(0) = \mathbf{0}$, $C'(a) \neq \mathbf{0}$, $\operatorname{rank}(H'(C(a))) = N+3$, and

$$H(C(a)) = 0, \tag{32}$$

for all $a \in J$. Consider the solution curve $C(s) = (\mathbf{U}(s), \lambda(s))$ (parametrized for convenience with respect to arclength) such that $C(0) = (\mathbf{0},0)$. The solution curve $H^{-1}(\mathbf{U}(0), \lambda(0))$ should be either diffemorphic to the circle or to the real line. Since the solution point $(\mathbf{0},0)$ is unique for $\lambda = 0$, it follows that C cannot be closed, and hence, it is diffemorphic to the real line. Since C is smooth curve, so $\mathbf{U}(s)$ is bounded for $\lambda(s) \in [0,1]$. Moreover, the curve C reaches the level $\lambda = 1$ after a finite arclength s_0, i.e., $C(s_0) = (\overline{\mathbf{U}}, 1)$, and hence, $\overline{\mathbf{U}}$ is the zero point of W. Thus, we can take J to be $[0,1]$. This choice for J was suggested by Smale. For more details about the proof, see [3].

Now, we want to explain how can we apply these ideas numerically. By differentiating equation (32) it follows that $C'(0)$ satisfies the equation

$$H'(C(0))C'(0) = 0,$$

and hence, $C'(0)$ is orthogonal to all rows of $H'(C(0))$. Then, for all $a \in [0,1]$, $\det[\begin{smallmatrix} H'(C(a)) \\ C'(a)^T \end{smallmatrix}] > 0$, $\|C'(a)\| = 1$, and $H'(C(a))C'(a) = 0$, where $\|\cdot\|$ means the Euclidean norm. We will use the predictor-corrector technique to numerically trace the curve C. We will use the Euler-predictor which is given by

$$\mathbf{X} = \mathbf{Z} + ht(H'(\mathbf{Z})), \tag{33}$$

where $\mathbf{Z}$ is a point lying along the curve C, and $h > 0$ represents a fixed stepsize, as a predictor step. The corrector we will use is called the Gauss-Newton-corrector which is given by

$$\mathbf{Y} = \mathbf{X} - H'(\mathbf{X})^{+} H'(\mathbf{X}). \tag{34}$$

For more details about the predictor-corrector technique, see [4]. It easy to see that $H'(\mathbf{U},\lambda) = [M + \lambda \frac{d}{d\mathbf{U}}(A\mathbf{U} \odot B\mathbf{U})A\mathbf{U} \odot B\mathbf{U} - R]$, where $\frac{d}{d\mathbf{U}}(A\mathbf{U} \odot B\mathbf{U})$ can be computed using equation (8). We will start from $\mathbf{Z}_0 = (\mathbf{0}, 0)$ and then we generate the sequence $\mathbf{Z}_0, \mathbf{Z}_1, \mathbf{Z}_2, \ldots$. We stop our tracing of the curve C at $\mathbf{Z}_k$ if the last component of $\mathbf{Z}_k$ is less than or equal to 1 and the last component of $\mathbf{Z}_{k+1}$ is greater than 1. In this case, we can rewrite $\mathbf{Z}_k$ as $\mathbf{Z}_k = (\mathbf{U}_k, \lambda_k)$. Thus, $\mathbf{U}_k$ will be the approximate solution of the nonlinear system (29). Also, we substitute $\mathbf{U}_k$ in equation (9) to get an approximation for the solution of problem (1),(2). We summarize our discussion in the following algorithm.

ALGORITHM 1.
Input: Positive integer N stepsize $h > 0$.
Output: Approximate solution $V_N(x)$ for problem (1),(2).

Step 1: Compute the matrices B_1, B_2, B_3 and B_4.
% use equations (17)–(20).
Step 2: Compute the matrices A_1 and A_2. % use equations (6) and (7).
Step 3: Set $Q_1 = BA_2 + B_4$, $Q_3 = B_3 A_1$.
Step 4: Compute the matrices M, A, B and $\mathbf{R}$. % use equation (29).
Step 5: Set $\mathbf{Z} = [\mathbf{0}0] \in \Re^{N+4}$ and $\mathbf{Y} = Z$.
Step 6: While $\mathbf{Z}_{N+4} < 1$, do steps 7–9.
Step 7: Set $\mathbf{Z} = \mathbf{Y}$.
Step 8: Predict a point $\mathbf{X}$. % use equation (33).
Step 9: Use the corrector to find $\mathbf{Y}$. % use equation (34).
Step 10: Set $\mathbf{U}$ to be the first $N + 3$ components of $\mathbf{Z}$.
Step 11: Compute $V_N(x)$. % Use equation (9).
Step 12: Stop.

4. NUMERICAL RESULTS

In this section, we present two of our experimental examples. Then, we state some conclusions about the results. All calculations are carried out using the 586 IBM computer. Programs are written in double precision.

Throughout our examples, we used the following notation:

N : the number of terms in the approximate solution in equation (9);
h : the step size;
ϵ_N: the error in the approximate solution using the technique of this paper which is given by,

$$\epsilon_N = \max\{\, |v_{\text{exact}}(x) - v_N(x)| : x \in \{-1, -0.99, \ldots, 0.99, 1\}\};$$

$\overline{\epsilon_N}$: the error in the approximate solution using syam's technique which is given by,

$$\overline{\epsilon_N} = \max\{|v_{\text{exact}}(x) - v_N(x)| : x \in \{-1, -0.99, \ldots, 0.99, 1\}\};$$

T_N: the computer time which is needed to compute the approximate solution using the technique of this paper;
$\overline{T_N}$: the computer time which is needed to compute the approximate solution using Syam's technique.

EXAMPLE 1. Consider

$$V''(x) + xe^{-x}V(x)V'(x) + (1 + x^2)V(x) = (2 + x + x^3)e^x, x \in]-1, 1[,$$
$$V(-1) = e^{-1} \quad \text{and} \quad V(1) = e.$$

Then, the exact solution is $V(x) = e^x$. In Table 1, we will study the relation between the number of terms in the approximate solution N and the error in the approximate solution ϵ_N using the stepsize $h = 0.001$. Also, the execution time will be presented for each N.

Table 1.

N	ϵ_N	T_N	$\overline{\epsilon_N}$	$\overline{T_N}$
6	1.231e^{-7}	1.0	1.557e^{-3}	3.1
8	3.453e^{-9}	1.2	2.158e^{-4}	5.0
10	3.234e^{-11}	1.8	1.121e^{-5}	6.1
12	7.216e^{-12}	2.3	1.218e^{-6}	8.2
14	4.629e^{-14}	2.6	8.710e^{-7}	10.0
16	1.213e^{-16}	3.0	2.189e^{-7}	11.7

EXAMPLE 2. Consider

$$V''(x) + \sin(\pi x)V(x)V'(x) - \pi\cos(\pi x)V(x) = f(x), x \in]-1, 1[,$$
$$V(-1) = 0 = V(1),$$

where $f(x) = \pi\sin(\pi x)(-\pi + (1/2)\sin(2\pi x) - \cos(\pi x))$. Then, the exact solution is $V(x) = \sin(\pi x)$. We make an entirely analogous analysis to that of example (1). We present our results in Table 2.

Table 2.

N	ϵ_N	T_N	$\overline{\epsilon_N}$	$\overline{T_N}$
6	2.310e^{-8}	1.1	1.0e^{-2}	3.2
8	3.752e^{-10}	1.3	2.1e^{-3}	5.1
10	5.134e^{-11}	1.9	4.1e^{-4}	6.2
12	9.261e^{-13}	2.4	2.3e^{-5}	8.1
14	3.745e^{-14}	2.8	1.1e^{-6}	10.1
16	4.729e^{-16}	3.0	2.1e^{-7}	11.6

From the results in Tables 1 and 2 we can make the following remarks.

1. Our approach in this paper, works nicely and efficiently.
2. The error in our approach is smaller than the error of Syam's approach [1].
3. The computer time in our approach is less than the computer time of Syam's approach [1].

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